

GEOMETRIC NULLSTELLENSATZ AND SYMBOLIC POWERS ON ARBITRARY VARIETIES

WENBO NIU

ABSTRACT. In recent years, a multiplier ideal defined on arbitrary varieties, so called Mather multiplier ideal, has been developed independently by Ein-Ishii-Mustata, and de Fernex-Docampo. With this new tool, we have a chance of extending some classical results proved in nonsingular case to arbitrary varieties to establish their general forms. In this paper, we first extend a result of geometric nullstellensatz due to Ein-Lazarsfeld in nonsingular case to any projective varieties. Then we prove a result on comparison of symbolic powers with ordinary powers on any varieties, which extends results of Ein-Lazarsfeld-Smith and Hochster-Huneke.

1. INTRODUCTION

Throughout this paper, for simplicity, we work over the complex number field $k := \mathbb{C}$. A scheme is always of finite type over k and a variety is a reduced irreducible scheme.

Our first result extends a result of geometric nullstellensatz due to Ein and Lazarsfeld [EL99]. For the motivation and background about geometric nullstellensatz we refer to this paper where the authors give a very detailed introduction on this area and related works.

Theorem 1.1. *Let X be a projective variety of dimension n with an ample line bundle L . Let \mathfrak{j}_X be the Jacobian ideal of X and \mathfrak{a} an ideal sheaf of \mathcal{O}_X . Assume that $\mathfrak{a} \otimes L^d$ is globally generated for some positive integers d . Then one has*

$$\mathfrak{j}_X \cdot (\sqrt{\mathfrak{a}})^{n \cdot d^n \cdot \deg_L X} \subseteq \mathfrak{a}.$$

For the Jacobian ideal, if $\mathfrak{j}_X \otimes L^\delta$ is globally generated for some positive integer δ , then one has

$$(\sqrt{\mathfrak{j}_X})^{(n+1) \cdot \delta^n \cdot \deg_L X} \subseteq \mathfrak{j}_X.$$

The proof the theorem follows the spirit of [EL99] utilizing Mather multiplier ideals instead of usual multiplier ideals. Roughly speaking, we insert the Jacobian ideal into the formula obtained by Ein and Lazarsfeld. The above result is certainly not optimal but works for any projective varieties. However, if we put some singularities conditions on varieties, then we can remove the Jacobian ideal in the formula (see Remark 3.2 and Corollary 3.3).

Our second result gives a comparison of symbolic powers with ordinary powers for a radical ideals on arbitrary varieties. Let X be a variety and Z be a proper reduced subscheme of X defined by an ideal sheaf \mathfrak{q} . Assume that every component of Z has codimension $\leq e$. The t -th symbolic power is then defined by $\mathfrak{q}^{(t)} = \{f \in \mathcal{O}_X \mid f \in \mathfrak{m}_\eta^t, \text{ for all generic point } \eta \text{ of } Z\}$, where \mathfrak{m}_η means the maximal ideals in the local ring $\mathcal{O}_{X,\eta}$. When X is nonsingular, a surprising result of Ein, Lazarsfeld and Smith [ELS01] shows that $\mathfrak{q}^{(me)} \subseteq \mathfrak{q}^m$ for any $m \geq 1$. Based on tight closure theory and reduction to characteristic p , this result has been generalized by Hochster and Huneke [HH02] and in particular when X is singular they showed that $\mathfrak{j}_X^{m+1} \cdot \mathfrak{q}^{(me)} \subseteq \mathfrak{q}^m$

2010 *Mathematics Subject Classification.* 13A10, 14Q20.

Key words and phrases. Geometric nullstellensatz, symbolic powers.

where j_X is the Jacobian ideal of X .¹ In their paper, Hochster and Huneke also asked if one can drop the power of j_X^{m+1} by 1 to be j_X^m [HH02, 5. Questions]. Now by using Mather multiplier ideals and inspired by the pioneer work of Ein, Lazarsfeld and Smith, we are able to give a geometric approach to this question and drop the power of the Jacobian ideal as expected.

Theorem 1.2. *Let X be a variety and Z be a proper reduced subscheme of X defined by an ideal sheaf \mathfrak{q} . Assume that every component of Z has codimension $\leq e$. Then for any $m \geq 1$,*

$$j_X^m \cdot \mathfrak{q}^{(me)} \subseteq \mathfrak{q}^m,$$

where j_X is the Jacobian ideal of X .

The basic idea to prove this theorem is to use asymptotic Mather multiplier ideals which generalizes asymptotic multiplier ideals on nonsingular varieties used in [ELS01]. Of course, some technical difficulties should be cleared in order to get the desired result. At this point, we thank the paper [EIM11] for providing two important basic theorems: local vanishing theorem and subadditivity theorem for Mather multiplier ideals. Now only one main difficulty left to us is to relate asymptotic Mather multiplier ideals of symbolic powers to ordinary powers. For certain generalization of the above theorem, see Remark 4.2.

This paper is organized as follows. In the section 2, we briefly review the theory of Mather multiplier ideals and its asymptotic construction. In the section 3, we prove Theorem 1.1 and the section 4 is devoted to Theorem 1.2.

2. MATHER MULTIPLIER IDEALS

In this section, we briefly recall the theory of Mather multiplier ideals developed by Ein, Ishii and Mustata in a recent paper [EIM11] (see also the work of de Fernex and Docampo [dFD12]). Some standard definitions are quoted directly from the paper for the convenience of the reader. Then we outline a basic properties of asymptotic version of Mather multiplier ideals.

Let X be an n -dimensional variety and let Ω_X be the sheaf of differentials of X . Consider the morphism

$$\pi : \mathbb{P}(\wedge^n \Omega_X) \longrightarrow X$$

which is an isomorphism over the nonsingular locus X_{reg} of X . The *Nash blow-up* \widehat{X} is the closure of $\pi^{-1}(X_{\text{reg}})$ in $\mathbb{P}(\wedge^n \Omega_X)$ with the reduced scheme structure. The *Jacobian ideal* j_X of X is defined to be the n -th Fitting ideal of Ω_X , i.e., $j_X := \text{Fitt}^n \Omega_X$.

Given several ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ on X , a *log resolution of singularities* of X and $\mathfrak{a}_1 \cdots \mathfrak{a}_t$ is a projective birational morphism $f : Y \longrightarrow X$, with Y nonsingular such that for each i , $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$ for an effective divisor F_i on Y and the union of $\text{Exc}(f)$ with those F_i 's are simple normal crossings.

Definition 2.1. Let X be a variety of dimension n and $f : Y \longrightarrow X$ be a resolution of singularities factoring through the Nash blow-up of X . Then the image of the canonical homomorphism

$$f^*(\wedge^n \Omega_X) \longrightarrow \wedge^n \Omega_X$$

is an invertible sheaf of the form $J \cdot \wedge^n \Omega_X$ where J is the invertible ideal sheaf on Y that defines an effective divisor which is called the *Mather discrepancy divisor* and denoted by $\widehat{K}_{Y/X}$.

¹Hochster and Huneke considered more general case in the setting of commutative algebra. However, here we stay in our geometric setting of this problem.

Definition 2.2. Let X be a variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ a nonzero ideal on X . Given a log resolution $f : Y \rightarrow X$ of X and $j_X \cdot \mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and $j_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$ for some effective divisors Z and $J_{Y/X}$ on Y (such resolution automatically factors through the Nash blow-up, see Remark 2.3 of [EIM11]). The *Mather multiplier ideal* of \mathfrak{a} of exponent $t \in \mathbb{R}_{\geq 0}$ is defined by

$$\widehat{\mathcal{I}}(X, \mathfrak{a}^t) := f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ \rfloor),$$

where $\lfloor \cdot \rfloor$ means the round down of an \mathbb{R} -divisor.

Remark 2.3. (1) Here we stick to the terminology used in [EIM11]. A similar theory also developed independently by de Fernex and Docampo in [dFD12], where they use a different terminology, namely Jacobian multiplier ideal.

(2) If the variety X is nonsingular, then the Mather multiplier ideal is simply the classical multiplier ideal. But these two notions are not necessarily equal on a \mathbb{Q} -Gorenstein variety (cf. [EIM11] and [dFD12]), where both of them are defined.

Now we state the asymptotic construction of Mather Multiplier ideals. For asymptotic multiplier ideals on nonsingular varieties we refer to the book [Laz04] or the paper [ELS01]. Since the theory now has become very classical we shall be brief here.

Definition 2.4. A graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \geq 1}$ on a variety X is a collection of nonzero ideal sheaves $\mathfrak{a}_k \subseteq \mathcal{O}_X$ satisfying $\mathfrak{a}_i \cdot \mathfrak{a}_j \subseteq \mathfrak{a}_{i+j}$ for all $i, j \geq 1$.

There are some examples of graded family can be found in [ELS01, Example 1.2]. In particular, the collection of symbolic powers of a radical ideal is naturally a graded family of ideals.

Fix an index $l \geq 1$ in a graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \geq 1}$ and fix a positive real number $c > 0$. Consider the set of Mather multiplier ideals

$$(2.4.1) \quad \left\{ \widehat{\mathcal{I}}\left(X, \frac{c}{p} \mathfrak{a}_{pl}\right) \right\}_{p \geq 1}$$

It is standard to check (cf. [ELS01, Lemma 1.3.] or the last paragraph of [EIM11]) that there is a unique maximal element in the set (2.4.1), which is the so called asymptotic Mather multiplier ideals.

Definition 2.5. Given a graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \geq 1}$ on a variety X , the *asymptotic Mather multiplier ideal* at level l associated to $c > 0$ is the maximal element of the set (2.4.1), and is denoted by $\widehat{\mathcal{I}}(X, c \cdot \|\mathfrak{a}_l\|)$. It is clear that for $p \gg 0$, $\widehat{\mathcal{I}}(X, c \cdot \|\mathfrak{a}_l\|) = \widehat{\mathcal{I}}(X, \frac{c}{p} \mathfrak{a}_{pl})$.

The following proposition describes basic properties of asymptotic Mather multiplier ideals and extends the core technical part of [ELS01].

Proposition 2.6. Let X be a variety and $\mathfrak{a}_\bullet = \{\mathfrak{a}_l\}_{l \geq 1}$ be a graded family of ideals. Let j_X be the Jacobian ideal of X . Then one has

- (1) for any $l \geq 1$, $j_X \cdot \mathfrak{a}_l \subseteq \widehat{\mathcal{I}}(X, \|\mathfrak{a}_l\|)$;
- (2) for any $m \geq 1$,

$$j_X^{m-1} \cdot \widehat{\mathcal{I}}(X, \|\mathfrak{a}_{ml}\|) \subseteq \widehat{\mathcal{I}}(X, \|\mathfrak{a}_l\|)^m;$$

- (3) suppose that $\mathfrak{b} \subseteq \mathcal{O}_X$ is an ideal such that $\widehat{\mathcal{I}}(X, \|\mathfrak{a}_l\|) \subseteq \mathfrak{b}$ for some fixed index $l \geq 1$, then for any $m \geq 1$,

$$j_X^m \cdot \mathfrak{a}_{ml} \subseteq \mathfrak{b}^m.$$

Proof. For (1), it is enough to show $j_X \cdot \mathfrak{a}_l \subseteq \widehat{\mathcal{J}}(X, \mathfrak{a}_l)$. Take a log resolution $f : Y \rightarrow X$ of $j_X \cdot \mathfrak{a}_l$ such that $j_X \cdot \mathcal{O}_X = \mathcal{O}_X(-J_{Y/X})$ and $\mathfrak{a}_l \cdot \mathcal{O}_X = \mathcal{O}_X(-Z_l)$ where $J_{Y/X}$ and Z_l are effective divisors on Y . Since $\widehat{K}_{Y/X}$ is effective we then have

$$j_X \cdot \mathfrak{a}_l \subseteq \overline{j_X \cdot \mathfrak{a}_l} \subseteq f_* \mathcal{O}_Y(-J_{Y/X} - Z_l) \subseteq f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - Z_l) = \widehat{\mathcal{J}}(X, \mathfrak{a}_l),$$

where $\overline{\tau}$ means the integral closure of an ideal τ .

For (2), let $p \gg 0$ such that

$$\widehat{\mathcal{J}}(X, \|\mathfrak{a}_l\|) = \widehat{\mathcal{J}}(X, \frac{1}{pm} \mathfrak{a}_{pml}), \text{ and } \widehat{\mathcal{J}}(X, \|\mathfrak{a}_{ml}\|) = \widehat{\mathcal{J}}(X, \frac{1}{p} \mathfrak{a}_{pml}).$$

We can rewrite

$$\widehat{\mathcal{J}}(X, \frac{1}{p} \mathfrak{a}_{pml}) = \widehat{\mathcal{J}}(X, \frac{m}{pm} \mathfrak{a}_{pml}) = \widehat{\mathcal{J}}(X, (\mathfrak{a}_{pml}^{\frac{1}{pm}})^m).$$

Now we have

$$\begin{aligned} j_X \cdot \widehat{\mathcal{J}}(X, (\mathfrak{a}_{pml}^{\frac{1}{pm}})^m) &\subseteq \widehat{\mathcal{J}}(X, j_X \cdot (\mathfrak{a}_{pml}^{\frac{1}{pm}})^m) = \widehat{\mathcal{J}}(X, j_X \cdot (\mathfrak{a}_{pml}^{\frac{1}{pm}})^{m-1} \cdot \mathfrak{a}_{pml}^{\frac{1}{pm}}) \\ &\subseteq \widehat{\mathcal{J}}(X, (\mathfrak{a}_{pml}^{\frac{1}{pm}})^{m-1}) \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}_{pml}^{\frac{1}{pm}}) \end{aligned}$$

where the left-hand-side inclusion is an immediate consequence of the definition of Mather multiplier ideals and the right-hand-side inclusion is from the Subadditivity Theorem [EIM11, Theorem 3.14]. Finally, (2) can be proved iteratedly.

(3) is a direct consequence of (1) and (2). \square

3. GEOMETRIC NULLSTELLENSATZ ON A PROJECTIVE VARIETY

In this section, based on Mather multiplier ideals we prove Theorem 1.1 and give some improved results by putting certain singularities conditions on varieties. Our approach follows the work of [EL99] and we refer the reader to it for further information.

We recall the distinguished subvarieties associated to an ideal sheaf. Let X be a variety and \mathfrak{a} be an ideal of \mathcal{O}_X . Let $\nu : W \rightarrow X$ be the normalization of the blowing-up of X along \mathfrak{a} such that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-E)$ where E is an effective Cartier divisor on W . We can write

$$E = \sum_{i=1}^t r_i E_i$$

as a sum of prime divisors E_i 's with some positive integer coefficients r_i . Write $Z_i = \nu(E_i)$ to be the image of E_i on X with the reduced scheme structure. Then Z_i 's are called the *distinguished subvarieties* of \mathfrak{a} with the coefficient r_i . The following proposition bounds the coefficients r_i 's of distinguished subvarieties in terms of global invariants.

Proposition 3.1 ([EL99, Proposition 3.1]). *Let X be a projective variety with an ample line bundle L . Let \mathfrak{a} be an ideal sheaf of \mathcal{O}_X such that $\mathfrak{a} \otimes L^d$ is generated by global sections. Then*

$$\sum_{i=1}^t r_i \cdot d^{\dim Z_i} (\deg_L Z_i) \leq d^{\dim X} (\deg_L X).$$

where Z_i 's are distinguished subvarieties of \mathfrak{a} .

Next we prove Theorem 1.1. The basic idea is essentially the same as the one in [EL99] so we shall be brief.

Proof of Theorem 1.1. Let Z_i for $i = 1, \dots, t$ be the distinguished subvarieties of \mathfrak{a} with the coefficient r_i defined by the ideal \mathfrak{q}_{Z_i} .

Claim 3.1.1. For $l \geq n$ one has the inclusion

$$\mathfrak{j}_X \cdot (\mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)}) \subseteq \widehat{\mathcal{F}}(X, \mathfrak{a}^l).$$

Proof of claim. Let $f : Y \rightarrow X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{a}$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ for some effective divisors $J_{Y/X}$ and Z on Y . For any element $g \in \mathfrak{j}_X \cdot (\mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)})$ we show that

$$(3.1.2) \quad \operatorname{div} f^* g \geq J_{Y/X} + lZ,$$

where $\operatorname{div} f^* g$ means the effective divisor defined by $f^* g$ on Y . To see this let $\nu : W \rightarrow X$ be the normalization of the blowing-up of X along \mathfrak{a} such that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-E)$ where E is an effective Cartier divisor on W . So we can write E as a sum of prime divisors, i.e., $E = \sum_{i=1}^t r_i E_i$ with the coefficients r_i 's. Notice that $Z_i = \mu(E_i)$ and f factors through ν via a morphism $\varphi : Y \rightarrow W$ such that $Z = \varphi^* E$. Now we can write $g = \sum b_i c_i$ as a finite sum where $b_i \in \mathfrak{j}_X$ and $c_i \in \mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)}$. It suffices to show $f^*(b_i c_i) \geq J_{Y/X} + lZ$ for each summand of g . Thus without loss of generality, we can assume that $g = bc$ where $b \in \mathfrak{j}_X$ and $c \in \mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)}$. Then it is clear that $\operatorname{ord}_{E_i} \nu^* c \geq r_i l$ and therefore $\operatorname{div} \nu^* c \geq lE$. Thus $\operatorname{div} f^* c = \operatorname{div} \varphi^*(\nu^* c) \geq \varphi^*(lE) = lZ$. On the other hand, it is clear $\operatorname{div} f^* b \geq J_{Y/X}$. Thus we conclude $\operatorname{div} f^* g \geq J_{Y/X} + lZ$.

Next, since $\widehat{K}_{Y/X}$ is effective we then have

$$\operatorname{div} f^* g + \widehat{K}_{Y/X} \geq J_{Y/X} + lZ.$$

Thus $g \in f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - lZ) = \widehat{\mathcal{F}}(X, \mathfrak{a}^l)$ as claimed. \square

Now by a Skoda-type formula [EIM11, Theorem 3.15], for $l \geq n$, $\widehat{\mathcal{F}}(X, \mathfrak{a}^l) \subseteq \mathfrak{a}$. On the other hand, if we set $r := \max_i \{r_i\}$, then it is clear that

$$(\sqrt{\mathfrak{a}})^{rl} \subseteq \mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)}.$$

Finally the value r is bounded by Proposition 3.1. The result then follows immediately if we take $l = n$.

For the Jacobian ideal, the proof is similar. This time let $Z_i, i = 1, \dots, t$, be the distinguished subvarieties of \mathfrak{j}_X with the coefficient r_i defined by the ideal \mathfrak{q}_{Z_i} . Then as the argument above, if set $r := \max_i \{r_i\}$ we can show that for $l \geq n + 1$

$$(\sqrt{\mathfrak{j}_X})^{rl} \subseteq \mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)} \subseteq \widehat{\mathcal{F}}(X, \mathfrak{j}_X^{l-1}) \subseteq \mathfrak{j}_X.$$

Then take $l = n + 1$ and use Proposition 3.1 to obtain the result. \square

Remark 3.2. (1) If X is nonsingular then \mathfrak{j}_X is trivial and the formula is essentially the one proved in [EL99].

(2) The above result works for arbitrary projective varieties but at the price that we have to insert the Jacobian ideal into the formula. It would be interesting to know if one can really remove the Jacobian ideal.

(3) If we put certain singularities conditions on the variety then we are able to remove the Jacobian ideal in the formula. As an example, we consider the following two typical singularities:

- (i) X is local complete intersection with canonical singularities, i.e., in a log resolution $f : Y \rightarrow X$ of X , the relative canonical divisor $K_{Y/X}$ is effective;
- (ii) X is J -canonical (cf. [EIM11] or [dFD12]), i.e., $\widehat{K}_{Y/X} - J_{Y/X}$ is effective in a log resolution $f : Y \rightarrow X$ of $\mathfrak{j}_X \cdot \mathfrak{a}$.

Then we can remove the Jacobian ideal and obtain the following corollary. The essential point is that under those singularities we can find an effective divisor on a resolution of singularities and it satisfies a local vanishing theorem (both singularities of (i) and (ii) are rational singularities) and induces a Skoda-type formula.

Corollary 3.3. *Let X be a projective variety of dimension n with an ample line bundle L and \mathfrak{a} be an ideal sheaf of \mathcal{O}_X . Assume that $\mathfrak{a} \otimes L^d$ is globally generated for a positive integers d and assume that X has the following singularities: (1) local complete intersection with canonical singularities; (2) J -canonical. Then one has*

$$(\sqrt{\mathfrak{a}})^{n \cdot d^n \cdot \deg_L X} \subseteq \mathfrak{a}.$$

Proof. Let us consider the case that X is a local complete intersection with canonical singularities. With the same notation as in the proof of Theorem 1.1 above, we can show that for $l \geq n$ one has the inclusion

$$\mathfrak{q}_{Z_1}^{(r_1 l)} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t l)} \subseteq \mathcal{J}(X, \mathfrak{a}^l),$$

where $\mathcal{J}(X, \mathfrak{a}^l)$ is the usual multiplier ideals. Now this time by singularities assumption, the relative canonical divisor $K_{Y/X}$ is effective and it has a local vanishing theorem since X has also rational singularities. Then simply by using a Skoda-type formula we see $\mathcal{J}(X, \mathfrak{a}^l) \subseteq \mathfrak{a}$. The rest is then the same as the proof of Theorem 1.1.

The case of J -canonical can be proved exactly in the same way but this time simply using the effective divisor $\widehat{K}_{Y/X} - J_{Y/X}$ instead of $K_{Y/X}$. \square

4. COMPARISON OF SYMBOLIC AND ORDINARY POWERS OF IDEALS

In this section, we proof the Theorem 1.2 on comparison of symbolic powers and ordinary powers of a radical ideal. As stated in Introduction, we basically follow the idea of [ELS01] but using asymptotic Mather multiplier ideals. The crucial point is to relate the asymptotic Mather multiplier ideals of symbolic powers to ordinary powers. In order to make the proof clean and transparent, we only deal with the basic case and leave the general case in a remark.

Let X be a variety and Z be a reduced subscheme defined by an ideal \mathfrak{q} . The collection of symbolic powers of \mathfrak{q} , $\{\mathfrak{q}^{(k)}\}_{k \geq 1}$, is a graded family of ideals. So for a fixed $l \geq 1$, we obtain the asymptotic Mather multiplier ideal $\widehat{\mathcal{J}}(X, \|\mathfrak{q}^{(l)}\|)$ at level l .

Lemma 4.1. *Let X be a variety and \mathfrak{q} be an ideal sheaf defining a proper reduced subscheme Z of X of codimension $\leq e$. Then*

$$\widehat{\mathcal{J}}(X, \|\mathfrak{q}^{(e)}\|) \subseteq \mathfrak{q}.$$

Proof. Let η be one generic point of Z . It suffices to show that $\widehat{\mathcal{J}}(X, \|\mathfrak{q}^{(e)}\|)_{\eta} \subseteq \mathfrak{q}_{\eta}$ since \mathfrak{q} is primary. Thus by working locally we can assume that Z is integral with only one generic point η . And we shall show that there is an open neighborhood of η on which $\widehat{\mathcal{J}}(X, \|\mathfrak{q}^{(e)}\|) \subseteq \mathfrak{q}$.

To this end, first of all, by taking an open neighborhood of η , we can assume that X is a subvariety of a nonsingular variety A of codimension c . Second, we can take an open set U in A containing η such that

- (1) $Z|_U$ is nonsingular;
- (2) If we write $U_X := X \cap U$ to be an open set of X , then on U_X the symbolic power $\mathfrak{q}^{(t)}$ is the same as the ordinary power \mathfrak{q}^t , i.e., for all $t \geq 1$, $\mathfrak{q}^{(t)}|_{U_X} = \mathfrak{q}^t|_{U_X}$.

(1) is clear because Z is generically nonsingular. (2) is because there are only finitely many embedded associated primes of \mathfrak{q}^t for all $t \geq 1$ so we can choose the open set U to avoid those primes which are certainly not \mathfrak{q} .

Finally, we replace A by this open set U . We denote by $\tilde{\mathfrak{q}}$ as the defining ideal of Z in A and by I_X as the defining ideal of X in A . Clearly, $I_X \subseteq \tilde{\mathfrak{q}}$ and $\tilde{\mathfrak{q}} \cdot \mathcal{O}_X = \mathfrak{q}$. By the construction above we see that $\widehat{\mathcal{J}}(X, ||\mathfrak{q}^{(e)}||) = \widehat{\mathcal{J}}(X, \mathfrak{q}^e)$ and thus in the following we shall show $\widehat{\mathcal{J}}(X, \mathfrak{q}^e) \subseteq \mathfrak{q}$.

Now let $\mu : A' := \text{Bl}_{\tilde{\mathfrak{q}}} A \rightarrow A$ be a log resolution of $\tilde{\mathfrak{j}}_X \cdot \tilde{\mathfrak{q}}$ where $\tilde{\mathfrak{j}}_X$ is the pull back of \mathfrak{j}_X in A , so that we have $\tilde{\mathfrak{q}} \cdot \mathcal{O}_{A'} = \mathcal{O}_{A'}(-E)$ and $\tilde{\mathfrak{j}}_X \cdot \mathcal{O}_{A'} = \mathcal{O}_{A'}(-J)$. Let X' be the strict transform of X in A' . Then by [EIM11, Lemma 2.11], we obtain a birational morphism $\varphi' : \bar{A} \rightarrow A'$ such that the composition morphism $\varphi := \mu \circ \varphi'$ is a factorizing resolution of X inside A (for definition of factorizing resolution see [EIM11, Definition 2.10]) and $\bar{X} \cup \text{Exc}(\varphi) \cup \text{Supp } \varphi'^* E \cup \text{Supp } \varphi'^* J$ has simple normal crossings, where \bar{X} is the strict transform of X in \bar{A} . The restriction morphism $\varphi|_{\bar{X}} : \bar{X} \rightarrow X$ then factors through the Nash blow-up of X . Furthermore, by the definition of factorizing resolution, we can write $I_X \cdot \mathcal{O}_{\bar{A}} = I_{\bar{X}} \cdot \mathcal{O}_{\bar{A}}(-R_{X/A})$ where $R_{X/A}$ is an effective divisor on \bar{A} supported on the exceptional locus of φ .

Claim 4.1.1. One has $\mathcal{O}_{\bar{A}}(-R_{X/A}) \subseteq \mathcal{O}_{\bar{A}}(-\varphi'^* E)$.

Proof of Claim. We write $\varphi'^* E = \sum r_i E_i$ as a sum of prime divisors of \bar{A} with certain coefficients. Since $I_X \subseteq \tilde{\mathfrak{q}}$ we have by our construction

$$(4.1.2) \quad I_{\bar{X}} \cdot \mathcal{O}_{\bar{A}}(-R_{X/A}) \subset \mathcal{O}_{\bar{A}}(-\sum r_i E_i).$$

Since E_i 's are simple normal crossings, we have $\mathcal{O}_{\bar{A}}(-\sum r_i E_i) = \cap \mathcal{O}_{\bar{A}}(-r_i E_i)$. Thus it is enough to show $\mathcal{O}_{\bar{A}}(-R_{X/A}) \subseteq \mathcal{O}_{\bar{A}}(-r_i E_i)$ for each E_i . We prove this at each closed point of \bar{A} . For this, let $y \in \bar{A}$ be a closed point and assume that $(\mathcal{O}_{\bar{A}, y}, \mathfrak{m}_y) = (B, \mathfrak{m})$ is the local ring at y . Then at the point y , we can assume that $\mathfrak{m} = (x_1, \dots, x_n)$ is generated by regular sequence where $n = \dim \bar{A}$, and since \bar{X} meets E_i as simple normal crossings, we can assume $I_{\bar{X}, y} = (x_1, \dots, x_c)$ and E_i is generated by x_{c+1} . Then the ideal $J := \mathcal{O}_{\bar{A}}(-r_i E_i)_y = (x_{c+1}^{r_i})$ is a (x_{c+1}) -primary ideal. Assume that $\mathcal{O}_{\bar{A}}(-R_{X/A})_y = (f)$ for some element $f \in B$. Then by 4.1.2, we see $x_1 \cdot f \in J$. But in any case no powers of x_1 can be contained in J thus f must be in J since J is a primary ideal. This shows that $\mathcal{O}_{\bar{A}}(-R_{X/A})_y \subseteq \mathcal{O}_{\bar{A}}(-r_i E_i)_y$. Thus we conclude that $\mathcal{O}_{\bar{A}}(-R_{X/A}) \subseteq \mathcal{O}_{\bar{A}}(-r_i E_i)$ and therefore the claim follows. \square

Now by [EIM11, Lemma 2.12] we have $K_{\bar{A}/A} - cR_{X/A}|_{\bar{X}} = \hat{K}_{\bar{X}/X} - J_{\bar{X}/X}$. Write $D = K_{\bar{A}/A} - cR_{X/A} - e\varphi'^* E$ and consider the exact sequence

$$0 \rightarrow I_{\bar{X}} \cdot \mathcal{O}_{\bar{A}}(D) \rightarrow \mathcal{O}_{\bar{A}}(D) \rightarrow \mathcal{O}_{\bar{X}}(D|_{\bar{X}}) \rightarrow 0.$$

As showed in [EIM11, Proof of Proposition 2.13] by using a spectral sequence argument we deduce that $R^1 \varphi_* I_{\bar{X}} \cdot \mathcal{O}_{\bar{A}}(D) = 0$. Also notice that $\varphi_* \mathcal{O}_{\bar{X}}(D|_{\bar{X}}) = \widehat{\mathcal{J}}(X, \mathfrak{q}^e)$. Thus pushing down this sequence via φ , we have a surjective morphism

$$\varphi_* \mathcal{O}_{\bar{A}}(D) \rightarrow \widehat{\mathcal{J}}(X, \mathfrak{q}^e) \rightarrow 0.$$

This means that $\varphi_* \mathcal{O}_{\bar{A}}(D) \cdot \mathcal{O}_X = \widehat{\mathcal{J}}(X, \mathfrak{q}^e)$. On the other hand, by Claim (4.1.1), we have

$$\mathcal{O}_{\bar{A}}(D) \subseteq \mathcal{O}_{\bar{A}}(K_{\bar{A}/A} - (c+e)\varphi'^* E).$$

Note that $\varphi_* \mathcal{O}_{\bar{A}}(K_{\bar{A}/A} - (c+e)\varphi'^* E) = \mathcal{J}(A, (c+e)Z)$, the usual multiplier ideal of Z . Since Z has codimension $\leq (c+e)$ in A , we then see that $\mathcal{J}(A, (c+e)Z) \subseteq \tilde{\mathfrak{q}}$. Thus $\varphi_* \mathcal{O}_{\bar{A}}(D) \subseteq \tilde{\mathfrak{q}}$ and therefore we conclude that $\widehat{\mathcal{J}}(X, \mathfrak{q}^e) \subseteq \tilde{\mathfrak{q}} \cdot \mathcal{O}_X = \mathfrak{q}$. \square

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.1, we have $\widehat{\mathcal{J}}(X, ||\mathfrak{q}^{(e)}||) \subseteq \mathfrak{q}$. Then apply Proposition 2.6 we immediately get the desired result. \square

Remark 4.2. We can generalize Theorem 1.2 to higher symbolic powers easily. We state the result here and leave the details to the reader.

(1) Lemma 4.1 can be generalized as follows: Let X be a variety and \mathfrak{q} be an ideal sheaf defining a proper reduced subscheme Z of X of codimension $\leq e$. Then for an integer $l \geq e$, $\widehat{\mathcal{J}}(X, ||\mathfrak{q}^{(l)}||) \subseteq \mathfrak{q}^{(l+1-e)}$. Consequently, we can generalize Theorem 1.2 as for an integer $l \geq e$, one has

$$\mathfrak{j}_X^m \cdot \mathfrak{q}^{(ml)} \subseteq (\mathfrak{q}^{(l+1-e)})^m,$$

for all $m \geq 1$.

(2) We also can consider unmixed ideals (cf. [ELS01, 3. Generalization]) to get the following result analogue to [ELS01, Variant.]. Let \mathfrak{q} be an unmixed ideal on a variety X , and assume that every associated subvariety of \mathfrak{q} has codimension $\leq e$. Then for an integer $l \geq e$, one has

$$\mathfrak{j}_X^m \cdot \mathfrak{q}^{(me)} \subseteq \mathfrak{q}^m,$$

for all $m \geq 1$.

REFERENCES

- [dFD12] Tommaso de Fernex and Roi Docampo. Jacobian discrepancies and rational singularities. 2012.
- [EIM11] Lawrence Ein, Shihoko Ishii, and Mircea Mustata. *Multiplier ideals via Mather discrepancy*. Arxiv, 2011.
- [EL99] Lawrence Ein and Robert Lazarsfeld. A geometric effective Nullstellensatz. *Invent. Math.*, 137(2):427–448, 1999.
- [ELS01] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. Uniform bounds and symbolic powers on smooth varieties. *Invent. Math.*, 144(2):241–252, 2001.
- [HH02] Melvin Hochster and Craig Huneke. Comparison of symbolic and ordinary powers of ideals. *Invent. Math.*, 147(2):349–369, 2002.
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907-2067, USA
E-mail address: niu6@math.purdue.edu